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ON CONSERVATION LAWS OF CONTINUUM MECHANICS

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Abstract—A unified variational formulation is advanced, leading to field equations and conservation laws in general mechanical continua. The formulation is applicable to dynamic processes in any medium which admits a Lagrangian. As a result of the procedure employed, physical (or canonical) momenta emerge on the same level as material momenta. The simplicity and transparency of the relations derived is a consequence of strict separation of descriptions in physical and material space which precludes the use of displacement as a field variable. One of the results shows that a distinction between field equations (or equations of motion) and conservation laws is no longer essential and all basic relations are, in fact, balance laws.

INTRODUCTION

To introduce the subject matter of this paper, it is not intended to supply a comprehensive review of related literature. Rather, merely those contributions will be briefly reviewed and referenced here, which seem to be directly relevant to the topic discussed.

Strong interest in imperfections of a crystal lattice within the physical theory of the solid state on one hand and the emergence of modeling such imperfections within the framework of the classical theory of elasticity on the other, has stimulated Eshebly[1] to introduce the concept of the force on an elastic singularity. The force is defined as the negative gradient of the total energy of the body under consideration with respect to the change in position of the defect within the body. Such an elastic singularity might be a dislocation, an inclusion, a vacancy, etc. and Eshelby has shown that the force on such a defect can be given as an integral over any surface enclosing it. In the absence of defects, the surface integral reduces to zero and thus embodies a conservation law.

Conservation laws for linear elastostatics without defects, based on Noether's theorem, were discussed in a more general framework by Günther[2]. He rededuced Eshelby's surface integral, which corresponds to translation invariance and found two additional conservation laws, corresponding to invariance of rotation and similarity (or scale change). Eshelby expanded his work on elasticity with defects in a series of papers[3–5]. They were concerned primarily with general theory, introduction of the energy-momentum tensor for this purpose and with more detailed studies of specific defects.

A quantity similar to the energy-momentum tensor for the problem of an elastic string was discussed also by Morse and Feshbach[6].

Research in plane fracture mechanics has led Rice[7], without knowledge of any earlier work quoted above, to represent Irwin's crack extension force (or energy release rate) as a path-independent integral (named by him the J-integral), which proved to be of great practical utility.

Independently of Günther[2], three types of conservation laws, both for linearized and finite elastostatics, were established by Knowles and Sternberg[8]. Budiansky and Rice[9] have indicated that these two additional integrals are associated with cavity rotation (they called it the L-integral) and cavity expansion (they called it the M-integral). The work of Knowles and Sternberg[8] was extended to linear elastodynamics by Fletcher[10] and some specific cases in elastodynamics and thermoelasticity were considered by G. Herrmann[11].

Clearly, quantities such as forces on defects (which have been also termed non-Newtonian or quasi-forces) must have features both in common and distinct from usual forces customarily employed in continuum mechanics.

In the first case, the forces are related to changes in position of "objects" with respect to the material in which they find themselves, while in the second, the forces are related to changes in position of material bodies (or particles) with respect to the physical space in which they are inserted.

The purpose of this paper, then, is to present a unified formulation leading to all known conservation laws of continuum mechanics. This derivation will be based on consequent use of field theories and consistent separation of physical and material space. It will be seen that the "old" conservation laws for a continuum, such as conservation of linear and angular momentum, are to be placed on the same footing as the "new" conservation laws, briefly described above. The mathematical vehicle to accomplish this task consists in a judicious application of a variational formalism in which the Lagrangian density plays a central role, as described in Section 1.

A re-examination of discrete and continuous systems in Section 2 and in particular the passage from the former to the latter, leads to the recognition of the strong requirement to distinguish throughout between physical and material coordinates. This requirement precludes the use of the displacement (or a related quantity such as stress) as a field variable, since, by definition, displacement involves both physical and material coordinates. (At a later stage, the results can be expressed in terms of the displacement and its derivatives.) As is shown in Section 3, this distinction is not identical here to the usual distinction between Eulerian and Lagrangian formulations. In the same section the variational principle of Section 1 is applied to a material continuum and a suitable choice of dependent and independent variables leads in Section 4, with utter simplicity, to both old and new conservation laws. The two sets are now clearly recognized as being conservation laws in physical and in material space, respectively, but either set can be represented in either set of coordinates. It is seen that the quantities "material momentum vector" and "material momentum tensor"† emerge in completely natural fashion in material space just as the usual "physical momentum vector" and "physical momentum tensor" (i.e. stress). The material quantities are entirely independent from the physical quantities and both sets are of equal "importance", the former describing the mechanics of objects (such as defects) with respect to material space, the latter the mechanics of objects (such as material bodies) with respect to physical space.

In order not to render this paper excessively long, not all admissible transformations have been discussed here. In particular, since the number of admissible transformations in material space is in general larger than in physical space, a detailed study of these properties will become essential. Another important aspect which will be discussed in detail in a later study concerns the symmetry of both material and physical momentum tensors. Especially the similarity transformation needs considerable elucidation. These considerations, as well as specific applications to fracture mechanics, will be presented in a sequel to this study.

1. MATHEMATICAL PRELIMINARIES BASED ON FIELD THEORY

We consider the action integral S given by

$$S = \int \mathrm{d}t \int \mathrm{d}^3\xi \mathscr{L}(\xi_i, t; \phi_i, \dot{\phi}_i, \phi_{i,j}) \tag{1}$$

where ξ_i are space-type coordinates, t the time, ϕ_i are the field variables and $\mathscr{L}$ is the Lagrangian, whose arguments are as indicated. The notation

$$\dot{\phi}_i = \frac{\partial\phi_i}{\partial t}; \qquad \phi_{i,j} = \frac{\partial\phi_i}{\partial\xi_j} \tag{2}$$

has been introduced. It is assumed that ξ_i and t are independent variables such that

$$\frac{\partial\xi_i}{\partial t} = 0; \qquad \frac{\partial t}{\partial\xi_i} = 0 \tag{3}$$

†In Eshelby's terminology, this is the energy-momentum tensor (without energy terms), the quasi-momentum or the non-Newtonian momentum.

with the range $i = 1, 2, 3$. This index i is, of course, generally not related to the index enumerating fields. In applications we have in mind here, however, both indices will have the same range 1–3. Among the four independent variables involved in S, t is considered here separately on purpose because of the special role it plays in many physical theories. Only in some theories such as electromagnetism or general relativity is it preferable to deal with a unified four-dimensional formulation in which the geometry of time-space places time on the same footing as other coordinates. Since in further development here we intend to deal with a non-relativistic approach to material systems, we prefer to keep the time coordinate distinct from the three others.

The variational Euler equations following from $\delta S = 0$ for the problem with fixed boundaries are

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}\right) + \frac{\partial}{\partial \xi_j}\left(\frac{\partial \mathcal{L}}{\partial \phi_{i,j}}\right) = \frac{\partial \mathcal{L}}{\partial \phi_i}. \tag{4}$$

Equations (4) are field equations or equations of motion for the fields ϕ_i associated with the action integral (1).

Conservation laws are usually based on certain groups of transformations of independent and dependent variables and a variational principle which has to be considered with variable boundaries, supplies both the equations of motion and conservation laws for the problem under consideration, this procedure being rather complicated.

We would like to propose here an alternate and simpler method to achieve the same goal. The principal idea takes advantage of the fact that the conservation laws may be understood as certain conditions, additional to the equations of motion, imposed on the Lagrangian. Thus these laws must lie latently buried already in the Lagrangian itself and it is the variation of action which brings them to the surface and makes them explicit. One may attempt, then, to make them explicit through a simpler operation, namely an operation on the Lagrangian itself, since from elementary mechanics it is known that certain conservation laws may be immediately recognized from the form of the Lagrangian. For example, if the Lagrangian does not depend explicitely on time, then energy is conserved. One of the simplest operations is differentiation with respect to independent variables. This is the procedure we shall follow here.

Differentiating $\mathcal{L}$ (as a function) with respect to ξ_i, we obtain

$$\begin{aligned}\mathcal{L}_i = \left(\frac{\partial \mathcal{L}}{\partial \xi_i}\right)_{\text{exp}} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j}\dot{\phi}_{j,i} + \frac{\partial \mathcal{L}}{\partial \phi_j}\phi_{j,i} + \frac{\partial \mathcal{L}}{\partial \phi_{j,i}}\phi_{j,ki} = \left(\frac{\partial \mathcal{L}}{\partial \xi_i}\right)_{\text{exp}} \\ + \frac{\partial}{\partial \xi_k}\left(\frac{\partial \mathcal{L}}{\partial \phi_{j,k}}\phi_{j,i}\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_j}\phi_{j,i}\right)\left[\frac{\partial \mathcal{L}}{\partial \phi_j} - \frac{\partial}{\partial \xi_k}\frac{\partial \mathcal{L}}{\partial \phi_{j,k}} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_j}\right)\right].\end{aligned}$$

If the equations of motion (4) are satisfied, the above expression simplifies to

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_j}\phi_{j,i}\right) + \frac{\partial}{\partial \xi_k}\left(\frac{\partial \mathcal{L}}{\partial \phi_{j,i}}\phi_{j,i} - \mathcal{L}\delta_{ik}\right) = -\left(\frac{\partial \mathcal{L}}{\partial \xi_i}\right)_{\text{exp}} \tag{5}$$

Similarly, differentiation with respect to time t results in

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}\dot{\phi}_i - \mathcal{L}\right) + \frac{\partial}{\partial \xi_i}\left(\frac{\partial \mathcal{L}}{\partial \phi_{j,i}}\dot{\phi}_j\right) = -\left(\frac{\partial \mathcal{L}}{\partial t}\right)_{\text{exp}} \tag{6}$$

Conservation laws in field theory are usually statements concerning quantities which are divergence-free. We observe that eqns (5) and (6) would become conservation laws if their right-hand sides would vanish, i.e. if the Lagrangian would not depend explicitely on the independent variables. In this sense, as we see, conservation laws express certain restrictions on the Lagrangian under consideration.

Equation (6) is related to possible transformations of time t. Physically, in classical mechanics, only one such transformation is admissible, namely translation of time. By contrast, many transformations are of interest with regard to space-type variables ξ_i. Possible other transformations will be discussed in[15] in the context of definite physical specifications of ξ_i.

2. DISCRETE AND CONTINUOUS SYSTEMS

The aim of this section is to show that the introduction of continuous material coordinates in place of a discrete number will not only result in changes of existing conservation laws for systems of particles, but creates potentially the possibility of obtaining a new class of conservation laws related to newly-introduced coordinates.

Let us examine how eqns (4)–(6) simplify for a single particle. Usually, in particle mechanics, independence of Lagrangian on time implies conservation of energy. Thus, from eqn (6), with $(\partial\mathscr{L}/\partial t)_{\exp}=0$, we will obtain the usual form of conservation of energy if ϕ_i is identified with the position of the particle x_i and the second term in eqn (6) disappears, since now $\mathscr{L}$ (and ϕ_i) do not depend on ξ_i.

Similarly, eqn (4) becomes a usual equation of motion, while in (5) each term becomes identically zero, because $\phi_{i,j}=0$ and $\partial\mathscr{L}/\partial\xi_i=0$.

Moreover, if in eqn (4) $\partial\mathscr{L}/\partial\phi_i\equiv\partial\mathscr{L}/\partial x_i$ is vanishing, we refer to it as a statement of conservation of momentum.

On the other hand, purely formally, since ξ_i was a space-type coordinate, ξ_i can be regarded as x_i, the position of the particle. From this point of view, eqns (4) and (5) coincide.

Reducing our formulation to a single particle provides physical interpretation of some derived equations; in our case eqn (4) can be interpreted as balance of momentum while (6) can be interpreted as balance of energy. Besides, we realize that on the level of discrete systems the analog of eqn (5) does not exist. This means that within the framework of a single variational principle we have the convenient formalism in which we can obtain both "old" (for discrete systems) and "new" (for continuous systems) conservation laws.

In anticipation of a fuller discussion in a later section, it might be remarked already at this point that the use of the displacement field u_i as ϕ_i is rather disadvantageous because u_i does not reduce, in the case of discrete systems, to the position coordinate of a particle.

3. VARIATIONAL PRINCIPLES FOR A MATERIAL CONTINUUM

In this section we wish to derive two distinct descriptions of motion of a material continuum and the relations between them. We shall employ the formalism presented in Section 2.

Let x_i be the physical coordinates (positions of material elements in a Newtonian reference frame) and X_i the material coordinates (i.e. either position in a reference configuration or the continualized number of a material element). The description of motion taking x_i as independent variables is often referred to as the Eulerian description, while that in which X_i are independent variables is termed the Lagrangian description. The displacement field $u_i=x_i-X_i$ is taken as the dependent variable in both descriptions. The distinction we intend to introduce below, however, goes further than the usual distinction between Eulerian and Lagrangian descriptions.

In our first description x_i and t shall be treated as the independent variables, while X_i shall play the role of fields ϕ_i. In this description the action S will be expressed by

$$S=\int \mathrm{d}t\int \mathscr{L}(x_i,t;X_k,V_k,X_{i;j})\,\mathrm{d}^3x \tag{7}$$

where

$$V_i=\frac{\partial X_i}{\partial t};\quad \frac{\partial X_i}{\partial x_j}=X_{i;j}. \tag{8}$$

Let us note that V_i is not the velocity of a material element. V_i can be interpreted as the rate of flow of material at a given point in space x_i, or briefly, material velocity. Further

$$\frac{\partial x_i}{\partial t}=0\quad\text{and}\quad\frac{\partial t}{\partial x_i}=0 \tag{9}$$

since they are independent variables.

In the second description the role of x_i and X_i is merely interchanged. The action S is now

$$S=\int \mathrm{d}t\int L(X_k,t;x_i,v_i,x_{i,j})\,\mathrm{d}^3X \tag{7'}$$

where

$$v_i = \frac{dx_i}{dt}; \quad \frac{\partial x_i}{\partial X_j} = x_{i,j} \tag{8'}$$

and

$$\frac{dX_i}{dt} = 0 \quad \text{and} \quad \frac{\partial t}{\partial X_i} = 0. \tag{9'}$$

In both descriptions we use the time t as an independent variable. Thus the symbol $\partial/\partial t$ is not uniquely defined because once x_i is held constant while in the second description X_i is held constant. To avoid this ambiguity we introduced in (8') the notation d/dt which is defined as

$$\left.\frac{\partial}{\partial t}\right|_{\bar{X}=\text{const.}} = \frac{d}{dt}. \tag{10}$$

This notation is consistent with traditional usage. Thus v_i denotes the velocity of the material element X_i.

It now becomes clear why the two descriptions introduced here cannot be termed merely as Lagrangian and Eulerian, respectively.

Comparing (7) and (7') we observe that not only have independent variables changed, but the independent variables as well. For this reason the Lagrangian will be different in the two cases, which is stressed by using different symbols $\mathscr{L}$ and L. We emphasize once more that in traditional formulations the displacement u_i is used as the dependent variable in both representations.

Once the action integral is given, the equations of motion and conservation laws can be written down.

In the first description they are

$$\frac{\partial}{\partial t} B_i + \frac{\partial}{\partial x_k} B_{ik} = \frac{\partial \mathscr{L}}{\partial X_i} \tag{A}$$

$$\frac{\partial}{\partial t} P_i + \frac{\partial}{\partial x_k} P_{ik} = -\left(\frac{\partial \mathscr{L}}{\partial x_i}\right)_{\text{exp}} \tag{B}$$

$$\frac{\partial}{\partial t} E + \frac{\partial}{\partial x_k} E_k = -\left(\frac{\partial \mathscr{L}}{\partial t}\right)_{\text{exp}} \tag{C}$$

where the following abbreviations have been employed

$$B_i = \frac{\partial \mathscr{L}}{\partial V_i} \quad B_{ik} = \frac{\partial \mathscr{L}}{\partial X_{i;k}} \tag{11}$$

$$P_i = \frac{\partial \mathscr{L}}{\partial V_j} X_{j;i} \quad P_{ik} = \frac{\partial \mathscr{L}}{\partial X_{j;k}} X_{j;i} - \mathscr{L}\delta_{ik} \tag{12}$$

$$E = \frac{\partial \mathscr{L}}{\partial V_i} V_i - \mathscr{L} \quad E_k = \frac{\partial \mathscr{L}}{\partial X_{j;k}} V_j. \tag{13}$$

We note that P_i and P_{ik}, as well as E and E_k can be expressed in terms of B_j and B_{jk}

$$\begin{aligned} P_i &= B_j X_{j;i} \quad P_{ik} = B_{jk} X_{j;i} - \mathscr{L}\delta_{ik} \\ E &= B_i V_i - \mathscr{L} \quad E_k = B_{jk} V_j. \end{aligned} \tag{14}$$

Without recognizing as yet the physical significance of these quantities, we observe that B_j and B_{jk} appear to be more basic than the other quantities appearing in the set of eqns (A)–(C).

In the second description we obtain the set of equations corresponding to (4)–(6), in that order

$$\frac{\mathrm{d}}{\mathrm{d}t} p_i + \frac{\partial}{\partial X_k} p_{ik} = \frac{\partial L}{\partial x_i} \tag{b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} b_i + \frac{\partial}{\partial X_k} b_{ik} = -\left(\frac{\partial L}{\partial X_i}\right)_{\mathrm{exp}} \tag{a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} e + \frac{\partial}{\partial X_k} e_k = -\left(\frac{\partial L}{\partial t}\right)_{\mathrm{exp}} \tag{c}$$

where the following abbreviations were introduced

$$p_i = \frac{\partial L}{\partial v_i} \quad p_{ik} = \frac{\partial L}{\partial x_{i,k}} \tag{15}$$

$$b_i = \frac{\partial L}{\partial v_j} x_{j,i} \quad b_{ik} = \frac{\partial L}{\partial x_{j,k}} x_{j,i} - L\delta_{ik} \tag{16}$$

$$e = \frac{\partial L}{\partial v_i} v_i - L \quad e_k = \frac{\partial L}{\partial x_{j,k}} v_j. \tag{17}$$

Again, we note that (16) and (17) can be expressed in terms of quantities employed in (15), i.e.

$$\begin{aligned} b_i &= p_j x_{j,i} \qquad b_{ik} = p_{jk} x_{j,i} - L\delta_{ik} \\ e &= p_i v_i - L \quad e_k = p_{jk} v_j. \end{aligned} \tag{18}$$

It is recalled once more that the difference between the two representations lies in a change of not only the iedependent variables but also of the dependent ones. The simplicity of the resulting equations is rather striking.

It would be possible to introduce a mixed representation in which only the independent variables would be changed. For example, instead of S given by (7′), let us consider

$$S = \int \mathrm{d}t \int \mathrm{d}^3x \underline{L}(x_i, t; X_j(x_i, t), v_k(x_i, t), x_{k,j}(x_i, t))$$

obtained from (7′) by making independent variables x_i dependent ones, but without changing the set of functions. Then, it can be easily shown that[12]

$$\frac{\mathrm{d}}{\mathrm{d}t} b_i + \frac{\partial}{\partial X_k} b_{ik} = j\left(\frac{\partial}{\partial t}\underline{b}_i + \frac{\partial}{\partial x_p}\underline{b}_{ip}\right) \tag{19}$$

where

$$\underline{b}_i = \frac{\partial \underline{L}}{\partial v_j} x_{j,i}, \quad \underline{b}_{ip} = v_p \frac{\partial \underline{L}}{\partial v_j} x_{j,i} + \frac{\partial \underline{L}}{\partial x_{j,k}} x_{j,i} x_{p,k} \tag{20}$$

and j is the Jacobian of the transformation, such that $L = j\underline{L}$. Comparing (20) with (16) we see that this representation leads to even more complicated forms for the material momenta than (16).

On the other hand, transition from (a) to (A) can be performed directly, providing not only considerable simplification (compare (16) with (11)), but making possible physical interpretation of obtained relations for the material momentum. The first term of (a) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_j} x_{j,i}\right) = -j\left(\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_k}(B_i v_k)\right)$$

while the others

$$\frac{\partial}{\partial X_k}\left[x_{j,i}\frac{\partial L}{\partial x_{j,k}} - L\delta_{ik}\right] = -j\frac{\partial}{\partial x_p}(-v_p B_i + B_{ip})$$

and finally

$$\frac{\mathrm{d}}{\mathrm{d}t}b_i + \frac{\partial}{\partial x_k}b_{ik} = -j\left(\frac{\partial}{\partial t}B_i + \frac{\partial}{\partial x_k}B_{ik}\right) \tag{21}$$

where B_i, B_{ik} are given by (11).

To derive the above relations (21), (19), we made use of some helpful formulas, like

$$\frac{\mathrm{d}j}{\mathrm{d}t} = j\frac{\partial v_k}{\partial x_k}, \quad \frac{\partial j}{\partial x_{i,k}} = \frac{\partial X_k}{\partial x_i}j$$

$$P_i = \frac{\partial \mathscr{L}}{\partial V_l}X_{l;i} \quad v_s = -V_p x_{s,p}, \text{ etc.}$$

4. EQUATIONS OF MOTION AND CONSERVATION LAWS

Since the set (A), (B), (C) and the set (a), (b), (c) have been derived from the same action integral S, one should expect that the information contained in each of the two sets is the same. Thus, relations between the two sets must exist which we would like to establish on a purely formal level first and then proceed to provide a physical interpretation. (The direct transition from (a) to (A) was shown in the preceding section.)

Comparing (C) and (c) we note that if $\mathscr{L}$ depends on t, then L will also and vice versa. Thus (C) and (c) correspond to each other (without being identical). Further, a similar observation can be made with regard to the dependence on X_i and consequently, (A) and (a) correspond to each other (see eqn 21).

Finally, the same argument concerning x_i leads to the conclusion that (B) and (b) represent the same law in two different (but equivalent) descriptions. We are led, however, to the remarkable observation that (B) was considered a conservation law, while (b) was the field equation.

The same holds true for (A) and (a). Thus we have to conclude that the designation of a relationship as an equation of motion or as a conservation law depends on the description chosen. Since both descriptions are equivalent, we shall use in the sequel the common term "balance laws".

It is to be noted that the relationship containing the explicit dependence of the Lagrangian on time in either description remains a conservation law. The reason for this is, of course, the fact that the time remains an independent variable in both descriptions. If we had introduced a four-dimensional quantity joining t and either x_i or X_i, we would have to combine E and E_k with B_j and B_{jk} in one description and e and e_k with p_j and p_{jk} in the ther description. But B_j, B_{jk} are not related to P_j, P_{jk} but rather to b_i, b_{ij}.

If an energy momentum tensor $T_{ik}(i, k = 1, \ldots, 4)$ were to be formed with E, E_k, B_j and B_{jk}, it would lead to a tensor t_{ij} in the other representation, whose components are made up of e, e_k, b_i, b_{ij}. On the other hand, in this representation we could have introduced, in a natural fashion, an energy momentum tensor t'_{ij} made up of components of e, e_k and p_j, p_{jk}. Thus we would confront two energy-momentum tensors in the same representation, which, however, would be rather strange, since some components would be identical, while others would be different. We prefer to renounce thus the rather delicate introduction of an energy-momentum tensor in classical continuum mechanics. Actually, the momentum-energy tensor is broadly used in relativistic dynamics of continuous media[13, 14] and is of crucial importance in Einstein's general relativity theory. It is related to (x_i, t) 4-space and its explicit form (see [14]) contains the velocity of light c, four-velocities and a relativistic stress tensor. The r.h.s. of Einstein's gravitational equations consists of the sum of two energy-momentum tensors for a material continuum and for the electromagnetic field. The reason to introduce the energy-momentum in relativistic dynamics of material continuum and in electromagnetism was the fact that transformation of space-time is coupled (by space we mean a physical space, not a material one) such that time and space components of corresponding fields in a natural way form 4-vectors or 4-tensors. In classical continuum mechanics, by contrast, the space coordinates do not mix with time.

We now approach the task of supplying a physical interpretation to the balance laws obtained.

We consider (C) and (c) and repeat that explicit independence of the Lagrangian on time in these equations is referred to as the expression of conservation of energy. Thus (C) and (c) should be interpreted as expressing the balance of energy. Hence E, defined in (13) and e, defined in (17) represent the energy density in physical and material space, respectively, while E_k and e_k (eqns 13 and 17) the corresponding energy fluxes.

Traditionally, the derivative of the Lagrangian with respect to the position of a material element x_i has been termed as a force acting on that element. Independence of the Lagrangian on position classically implies conservation of momentum. Thus (B) and (b) should be termed balance laws for linear momentum in the two descriptions, respectively. Thus P_i (defined in eqn 12) and p_i (defined in eqn 15) are momentum vectors, while P_{ik} and p_{ik} are momentum tensors.

Comparing eqns (12) and (15) we note that the definitions of p_i and p_{ik} are simpler, and thus more natural, than those of P_i and P_{ik}. If we consider linear elasticity, it is readily noted that p_{ik} corresponds to the stress tensor in the usual description, while P_{ik} is a different representation of the same quantity.

Next we turn our attention to eqns (A) and (a). We observe first that if we compare (A) with (B), B_i, B_{ik} are related to X_i in the same fashion as P_i, P_{ik} are related to x_i. This observation is confirmed for b_i, b_{ik} and p_i, p_{ik} in eqns (a), (b). For example, if $\mathscr{L}$ is independent of X_i, we obtain the conservation law for B_i, B_{ik} as we would get for p_i, p_{ik} if L did not depend on x_i. For this reason we shall call B_i the material momentum vector and B_{ik} the material momentum tensor.

We note the important fact that for a material continuum the material momenta B_i, B_{ik} (or b_i, b_{ik}) are just as important as the (physical) momenta p_i, p_{ik} (or P_i, P_{ik}) and are introduced into the general theory in a most natural way, once the general formalism adopted here is employed. Yet we realize that these material momenta have been dealt with but sparingly in the development of continuum mechanics and have been treated as somewhat extraneous artificial quantities. The reason for this was that in constructing a theory of continuum mechanics, the ideas successfully employed for systems of discrete particles were used. In such systems, material momenta do not exist. Further, early developments of continua were concerned with perfect materials, in which the material conservation laws are automatically satisfied. It is only when the interest in continua with defects arose, both on microscopic and macroscopic levels, was the material momentum introduced as an energy-momentum tensor without, however, placing both material and physical momenta into the same framework, as it is done here.

It is to be emphasized that the material momenta are independent of the physical momenta;† in particular, conservation of one does by no means imply conservation of the other.

The complete separation of material laws from physical laws presented in general form here, enhances to a large degree our potential understanding of the nature of the forces and their physical consequences. Unfortunately, previous developments in this area employed the displacement field as the dependent variable. Since this field and its derivatives depends on both material and physical coordinates, tracing of conservation laws in one or other descriptions was hardly possible.

5. CONCLUDING REMARKS

On the basis of a suitably chosen variational principle we were able to derive, and this in two different representations, not only the well-known balance laws for a material continua, but also additional balance-type laws involving the material momentum. The newly introduced quantities and laws appear as basic as the well-known physical momentum conservation laws. Whereas physical momentum conservation involves nonhomogeneities of physical space,

†In the same sense as, e.g. momenta are independent of energy.

produced by forces such as gravity, material momentum is related to nonhomogeneities of material space produced by defects.

In this paper the considerations were on quite a general level including dynamics and nonlinear effects. In subsequent papers we shall investigate the possibility of establishing further conservation laws in material space and relate them to path-independent integrals used in fracture mechanics.

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